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Mathematics

Homological versus algebraic equivalence in a jacobian

(algebraic cycle/integral/Fermat curve)

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ABSTRACT **Let Z be an algebraic p cycle homologous to zero in an algebraic complex manifold V. Associated to Z is a linear function ν on holomorphic $(2p + 1)$-forms on V, modulo periods, that vanishes if Z is algebraically equivalent to zero in V. I give a formula for ν for the case of V the jacobian of an algebraic curve C and Z = C − C′ (C′ = "inverse" of C) in terms of iterated integrals of holomorphic 1-forms on C. If C is the degree 4 Fermat curve, I use this formula to show that C − C′ is not algebraically equivalent to zero.**

Section 1. I consider the problem of deciding whether two homologous algebraic p-cycles ($p > 0$) C,C' in a compact nonsingular algebraic variety V over $\mathbf{C}$ are algebraically equivalent [i.e., roughly speaking, the cycle $C - C'$ can be "continuously" (algebraically) deformed into the cycle 0 (1)]. A direct method for proving algebraic *non*equivalence is to consider the linear function ν on holomorphic $(2p + 1)$-forms ω on V given by $\int_D \omega$, where D is a singular chain of topological dimension $2p + 1$ whose boundary is $C - C'$. If this linear function is not in the group of linear functions generated by integration over $(2p + 1)$-cycles, then C,C' are not algebraically equivalent [compare with the letter of W. Hodge (1951) quoted in ref. 2]. Here, I carry this out for the example of C, the degree 4 Fermat curve $F{:}x^4 + y^4 = 1$; V, its jacobian J; and C', the image of F under the automorphism of J given by the group–structure inverse. The problem reduces to the real number

$$\frac{2\displaystyle\int_0^1\left[\int_0^x \frac{dt}{(1-t^4)^{1/2}}\right]\frac{dx}{(1-x^4)^{3/4}}}{\left[\displaystyle\int_0^1 \frac{dt}{(1-t^4)^{1/2}}\right]\left[\displaystyle\int_0^1 \frac{dx}{(1-x^4)^{3/4}}\right]}. \qquad [1]$$

Because Eq. 1 is not an integer, $C = F$ is not algebraically equivalent to C' in $J(C)$. If Eq. 1 is irrational, then $C - C'$ would be of infinite order modulo algebraic equivalence. The method, based on ref. 3, is to give an explicit formula for $\int_D \omega$ (modulo periods of holomorphic 3-forms) in terms of iterated integrals on C of holomorphic 1-forms with periods in a subring of $\mathbf{C}$ and then specialize to curves whose normalized period matrix has entries in a quadratic imaginary field (gaussian numbers for F), in which case the subgroup of periods is *discrete* and so the integrals need only be numerically approximated.

This direct method has not been carried out previously. Instead, there is the method used by Griffiths (4) to give the first examples of algebraically nonequivalent homologous cycles that has been used in all subsequent papers on this subject—i.e., to consider families of varieties V_t and cycles C_t depending on the parameter t and differentiate with respect to t. The type of result thus obtainable is that algebraic nonequivalence holds for *generic* t (i.e., the complement of some countable union of subvarieties of the parameter space). In this way, Ceresa (5) recently proved algebraic nonequivalence of C,C' in $J(C)$ for generic C, while ref. 3 gives another way of doing this (nonvanishing of the differential of ν).

Section 2. Let C be a smooth complete curve over $\mathbf{C}$, p_0 be a base point, and $C \subset J(C)$ be the standard embedding with p_0 going to zero. I identify $H^1(C;\mathbf{Z})$, $H^1(J;\mathbf{Z})$, and the group $\mathcal{H}_{\mathbf{Z}}$ of real valued harmonic 1-forms (on C or J) with periods in $\mathbf{Z}$ and let $H^{1,0}$ [respectively, $H^{1,0}_{\mathbf{Z}(i)}$] denote holomorphic 1-forms (respectively, those with periods in the gaussian integers). Then, $H^3(J;\mathbf{Z}) = \Lambda^3(\mathcal{H}_{\mathbf{Z}})$ and I denote by $P_{\mathbf{Z}}$ the subgroup of primitive classes (those annihilated by the cup product with the Kähler form $g - 2$ times). $P_{\mathbf{Z}}$ is generated by elements $dh_1 \wedge dh_2 \wedge dh_3$, where $dh_i \in \mathcal{H}_{\mathbf{Z}}$ and are Poincaré duals in C of three *disjoint* simple closed curves on C (see ref. 3). We define a homomorphism $\nu{:}P \to \mathbf{R}/\mathbf{Z}$ associated to the cycle $C - C'$ on J by

$$\nu(dh_1 \wedge dh_2 \wedge dh_3) = \int_D dh_1 \wedge dh_2 \wedge dh_3 \ (\mathrm{mod}\ \mathbf{Z}),$$

where $\partial D = C - C'$. ν is the Abel–Jacobi image of $C - C'$ in the intermediate jacobian of J; it is independent of the choice of base point p_0 because I work with primitive cohomology.

For dh_i, $i = 1, 2, 3$, in $\mathcal{H}_{\mathbf{Z}}$ such that $\int_C dh_1 \wedge dh_2 = 0$, I define an iterated integral $I(dh_1,dh_2,dh_3)$ as follows (3): Let $\gamma \in \pi_1(C;p_0)$ represent a homology class Poincaré dual to dh_3; because $\int_C dh_1 \wedge dh_2 = 0$, let $dh_1 \wedge dh_2 = d\eta$ on C and η be a 1-form on C orthogonal to all closed 1-forms (and therefore unique). Then, $I(dh_1,dh_2,dh_3)$ will mean $\int_\gamma(h_1dh_2 - \eta)(\mathrm{mod}\ \mathbf{Z})$; considering γ as a path parametrized by $t \in [0,1]$, this last integral will mean

$$\int_{x=0}^1 \left\{\left[\int_{t=0}^x dh_1(t)\right] dh_2(x) - \eta(x)\right\} \mathrm{mod}\ \mathbf{Z}.$$

According to ref. 3,

$$\nu(dh_1 \wedge dh_2 \wedge dh_3) = 2I(dh_1,dh_2,dh_3)$$

if $dh_1 \wedge dh_2 \wedge dh_3$ is a generator of P as before. I can be considered as a homomorphism

$$(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})' \otimes \mathcal{H}_{\mathbf{Z}} \to \mathbf{R}/\mathbf{Z},$$

where $(\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}})'$ is the kernel of $\mathcal{H}_{\mathbf{Z}} \otimes \mathcal{H}_{\mathbf{Z}} \to \mathbf{Z}$ given by $dh_1 \otimes dh_2 \to \int_C dh_1 \wedge dh_2$.

I can now extend scalars from $\mathbf{Z}$ to $\mathbf{Z}(i)$ so that $\mathcal{H}_{\mathbf{Z}(i)} = \mathcal{H}_{\mathbf{Z}} \otimes \mathbf{Z}(i)$ = complex harmonic 1-forms with periods in $\mathbf{Z}(i)$ and $P_{\mathbf{Z}(i)} = P_{\mathbf{Z}} \otimes \mathbf{Z}(i)$. Then, $\nu{:}P_{\mathbf{Z}(i)} \to \mathbf{C}/\mathbf{Z}(i)$ is still given by $2I$, where $I{:}[\mathcal{H}_{\mathbf{Z}(i)} \otimes_{\mathbf{Z}(i)} \mathcal{H}_{\mathbf{Z}(i)}]' \otimes_{\mathbf{Z}(i)} \mathcal{H}_{\mathbf{Z}(i)} \to \mathbf{C}/\mathbf{Z}(i)$ and $\mathcal{H}_{\mathbf{Z}(i)}$ is isomorphic to $\pi_1(C;p_0)^{ab} \otimes \mathbf{Z}(i)$.

Assume now that $H^{1,0} = H^{1,0}_{\mathbf{Z}(i)} \otimes_{\mathbf{Z}(i)} \mathbf{C}$, where $H^{1,0}_{\mathbf{Z}(i)}$ denotes the holomorphic 1-forms with periods in $\mathbf{Z}(i)$. Further, let $g = 3$, so that $H^{1,0}$ has basis $\theta_1, \theta_2, \theta_3$ with the θ_i a $\mathbf{Z}(i)$-basis of $H^{1,0}_{\mathbf{Z}(i)}$. Then, ν restricted to holomorphic 3-forms is given by one complex number $\nu(\theta_1 \wedge \theta_2 \wedge \theta_3)\mathrm{mod}\ \mathbf{Z}(i)$ as follows: let $K_1,\ldots,K_6$

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be a $\mathbf{Z}$-basis of $H_1(C;\mathbf{Z})$ and $a_1, \ldots, a_6$ be corresponding elements of $\pi_1(C;p_0)$. Let the Poincaré dual of θ_3 be $\Sigma_1^6 \lambda_j K_j$, $\lambda_j \in \mathbf{Z}(i)$. Then,

$$\nu(\theta_1 \wedge \theta_2 \wedge \theta_3) = \sum_1^6 \lambda_j \int_{a_j} (\theta_1, \theta_2), \qquad [2]$$

where $\int_{a_j}(\theta_1, \theta_2)$ denotes the iterated integral $\int_0^1 [\int_0^x \theta_1]\theta_2$ (and $\eta = 0$ because $\theta_1 \wedge \theta_2 = 0$).

Section 3. Let $C = F{:}x^4 + y^4 = 1$. I will follow ref. 6: Let $\eta_{r,s,t} = x^{r-1}y^{s-4}dx$. Let

$$\eta_1^* = \eta_{1,1,2}/B\left(\frac{1}{4}, \frac{1}{4}\right),$$

$$\eta_2^* = \left(\frac{1-i}{2}\right)\eta_{1,2,1}/B\left(\frac{1}{4}, \frac{1}{2}\right),$$

$$\eta_3^* = \left(\frac{1-i}{2}\right)\eta_{2,1,1}/B\left(\frac{1}{2}, \frac{1}{4}\right),$$

where $B(r,s)$ is the β function $\dfrac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$. Projection to the x plane represents F as 4-sheeted covering branched over i^s, $s = 0, 1, 2, 3$. Make cuts in the x plane from i^s radially out to ∞, defining sheet s as that on which $y = i^s$ when $x = 0$. Choose as base point $(x = 1, y = 0)$, denoted 1. Let $a \in \pi_1(F;1)$ be the path from 1 to i on sheet 1 followed by i to 1 on sheet 0: $a = (1i)_1 \cdot (i1)_0$; this is the basic path used in ref. 6. Let K_1 be the homology class of a; then, the automorphisms $A(x,y) = (ix,y)$, $B(x,y) = (x,iy)$ give us the following canonical homology basis

$$K_1 = a,\ K_2 = Aa,\ K_3 = B^2a,\ K_4 = A^{-1}Ba,$$
$$K_5 = A^2B^2a,\ K_6 = A^2B^{-1}a, \qquad [3]$$

with intersection numbers $K_{2r-1} \circ K_{2r} = 1 = -K_{2r} \circ K_{2r-1}$, other $K_r \circ K_s = 0$.

Table 1. Periods of θ_j over k_r

	K_1	K_2	K_3	K_4	K_5	K_6
θ_1	$-i$	1	i	$-i$	$-i$	1
θ_2	0	0	$-(1+i)$	1	$1+i$	-1
θ_3	0	i	0	i	$1+i$	$i-1$

Let $\theta_1 = 2\eta_1^*$, $\theta_2 = (1-i)(\eta_2^* - \eta_1^*)$, $\theta_3 = (1-i)(\eta_3^* - \eta_1^*)$. Then, $\int_a \eta_j^* = -\frac{i}{2}$ and the periods of the θ_j over K_r are as given in Table 1. The θ_j are then a $\mathbf{Z}(i)$ basis of $H^{1,0}_{\mathbf{Z}(i)}$. The Poincaré dual of θ_2, defined by $PD(\theta_2) \circ K_j = \int_{K_j} \theta_2$, is

$$PD(\theta_2) = K_3 + (1+i)K_4 - K_5 - (1+i)K_6. \qquad [4]$$

We can now calculate $\nu(\theta_1 \wedge \theta_3 \wedge \theta_2)$ as the linear combination of iterated integrals (Eq. **2**) with coefficients λ_j from Eq. **4**, giving

$$\nu(\theta_1 \wedge \theta_3 \wedge \theta_2) = -16i \int_a (\eta_1^*, \eta_3^*)\ [\mathrm{mod}\mathbf{Z}(i)]. \qquad [5]$$

The integral [5] is transformed by the automorphism $\sigma(x,y) = (y/x\sqrt{i}, 1/xi)$ (and $\sigma\eta_3^* = \eta_2^*$, $\sigma\eta_2^* = -i\eta_1^*$, $\sigma\eta_1^* = i\eta_3^*$) into an integral from 0 to 1 on sheet 0 given by $-i$ times Eq. **1** if we take into account the β factors used to define η_j^*. Numerical calculation gives the value $[1.2435683 \pm 0.00576] \times (-i)$ for Eq. **5**. A more accurate value is $1.24178137867\ldots$.

Spencer Bloch has asked whether one could give examples for curves C defined over $\mathbf{Q}$. I wish to thank him, Ron Donagi, and Gerry Washnitzer for conversations.

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